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A UNIFORMLY VALID ASYMPTOTIC SOLUTION OF HART'S EQUATIONS FOR CONSTANT, NONELASTIC, EXTENSIONAL STRAIN RATE

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Abstract—A uniformly valid approximate solution of Hart's constitutive equation is presented in this paper for the special case of a constant, nonelastic strain rate tensile test. The method of matched asymptotics is used in the analysis. A principal result is that for sufficiently low temperature or high nonelastic strain rate, the "viscoplastic limit" is a good approximation to the solution of Hart's equations. The combined effect of temperature and strain rate on the behavior of these equations is shown to be characterized by two nondimensional parameters ϵ_1 and ϵ_2. It is found that with a judicious choice of these parameters, the numerical integration of Hart's equation can be easily carried out. The numerical results validate the viscoplastic approximation. A comparison of numerical results and the analytic solution are presented.

INTRODUCTION

The increasing demand for reliability of metallic structures has stimulated improvement of constitutive relations for the inelastic deformation of metals and alloys. This has led to the development of a phenomenological theory of deformation that is capable of describing the time and temperature dependence of plastic flow, in contrast to the time-independent description of classical plasticity.

The differential equations from these constitutive models are, in general, highly nonlinear. Closed-form solutions cannot be obtained even if the loading history and the geometry of the specimen considered are highly idealized. These difficulties hinder the interpretation and applicability of such theories. Instead of considering the general problem of integrating these equations for various theories, we shall restrict our considerations in the present study to the phenomenological theory proposed by Hart[1, 2]. So far, approaches to the integration problem have been of a purely numerical nature. Very few efforts have been made to investigate the analytical properties of these equations, their effects on the numerical methods or the dependence of the solution of these equations on different parameters. The purpose of this study is to investigate the integration problem by a detailed analysis of the constant, nonelastic strain rate problem using the approaches outlined above.

Hart's phenomenological model is summarized in the first section. The differential equation governing a given constant, nonelastic strain rate $\dot{\epsilon}$ is derived. Some of the numerical difficulties associated with this problem are briefly summarized. The idea of the "viscoplastic limit" is described, and its role in the numerical computation is discussed. In the second section, the governing equations are analyzed for the case of low temperature (or high strain rate) using the method of matched asymptotics. In the third section, numerical results are presented to further support the results of the second section.

The special case of constant, nonelastic strain rate $\dot{\epsilon} \equiv \dot{\epsilon}_{tot} - \dot{\sigma}/E$ (where $\dot{\epsilon}_{tot}$ is the strain rate and E is Young's modulus) analyzed in this study may be questioned. Constant $\dot{\epsilon}$ does not occur either in tests at constant total deformation rate $\dot{\epsilon}_{tot}$ or at constant stress rate $\dot{\sigma}$. In fact, constant $\dot{\epsilon}$ corresponds to a stress history that begins with a jump (to accommodate the elastic strain) and then continues to increase at a nonconstant rate. This deformation history thus has, perhaps, little direct relevance to common material tests. It is used in this work, however, because it simplifies the governing equations and leads to results that may possibly be generalized to a wider range of deformation histories.

EQUATION SUMMARY AND BACKGROUND

Hart's constitutive model as applied to the uniaxial case will be summarized in this section. We will briefly describe some of the numerical problems associated with these equations. The concept of viscoplastic limit will also be summarized below.

Hart's constitutive equations in the uniaxial case are relationships among the current values of the applied stress σ, the observable nonelastic strain rate $\dot{\epsilon}$, two state variables σ^* and a, the time rate of change of σ^* and a and the absolute temperature T. In Hart's formulation, the strain is the logarithmic strain, and the stress is the true stress. The equations also depend on various material constants, which will be described below. Physically, σ^* is interpreted as a measure of the current state of hardness of the material and a is a measure of the anelastic strain. These equations can be generalized to the multiaxial case as given by Hart[2].

The constraint conditions in Hart's model are

$$\dot{\epsilon} = \dot{a} + \dot{\alpha} \tag{1}$$

and

$$\sigma = \sigma_a + \sigma_f. \tag{2}$$

Equation (1) expresses the assumption that the total inelastic strain rate can be written as the sum of the anelastic strain rate $\dot{a}$ and the plastic strain rate $\dot{\alpha}$. Equation (2) expresses the assumption that the applied stress σ is the sum of two stress quantities σ_a and σ_f (the anelastic stress and the frictional stress). These stresses are defined by the following constitutive relationship with the strain or strain rate quantities a and $\dot{\epsilon}$, respectively, i.e.

$$\sigma_a = \mathcal{M}a \tag{3}$$

and

$$\sigma_f = \left[\frac{\dot{\epsilon}}{a^*(T)}\right]^{1/M}, \tag{4}$$

where $\mathcal{M}$ and M are positive material constants. The function $a^*(T)$ is determined by experiment. Typical values are $M \simeq 9$ and $\mathcal{M}$ of the order of G, the shear modulus, at temperature T. The plastic strain rate $\dot{\alpha}$ in eqn (1) is related to σ_a and the hardness state variable σ^* by

$$\dot{\alpha} = D\left(\frac{\sigma^*}{G}\right)^m \left(\ln\frac{\sigma^*}{\sigma_a}\right)^{-l}, \tag{5}$$

where m and l are positive material constants. In Hart's work[2], D is a material parameter given as

$$D = f\exp\left(\frac{-\phi}{RT}\right), \tag{6}$$

where R is the gas constant, ϕ the activation energy for atomic self-diffusion of the metallic species and f a material constant to be determined by experiments. The recovery term is not included in eqn (5) because its effects are negligible at low temperatures. Equation (6) implies that D is an extremely sensitive function of the absolute temperature T. Also, in Hart's paper[2], l is defined to be $1/\lambda$. Typical values of some of the constants are $m \approx 4$–5, $l \approx 6.6$.

To complete the constitutive model, the evolution of the hardness state variable σ^* is given by Hart[2]:

$$\dot{\sigma}^* = C\sigma^* \left(\frac{\sigma_a}{\sigma^*}\right)^k \left(\frac{G}{\sigma^*}\right)^m \dot{\alpha} \tag{7a}$$

$$= DC\sigma^* \left(\frac{\sigma_a}{\sigma^*}\right)^k \left(\ln \frac{\sigma^*}{\sigma_a}\right)^{-l}. \tag{7b}$$

The differential equations governing a constant, applied, nonelastic strain rate $\dot{\epsilon}$ can be obtained by combining eqns (1) and (3) with eqns (5) and (7b), i.e.

$$\dot{\sigma}^* \equiv \frac{d\sigma^*}{dt} = DC\sigma^* \left(\frac{\sigma_a}{\sigma^*}\right)^k \left(\ln \frac{\sigma^*}{\sigma_a}\right)^{-l} \tag{8}$$

$$\dot{\sigma}_a \equiv \frac{d\sigma_a}{dt} = \mathcal{M}\left[\dot{\epsilon} - D\left(\frac{\sigma^*}{G}\right)^m \left(\ln \frac{\sigma^*}{\sigma_a}\right)^{-l}\right]. \tag{9}$$

The solution of these two first-order nonlinear equations completely determines all the unknown variables, as σ_f can be calculated from the given strain rate $\dot{\epsilon}$ using eqn (4). The stress σ can then be calculated using eqn (2).

The initial value problem of $\sigma^*(t = 0) = \sigma_0^*$ and $\sigma_a(t = 0) = 0$ will be considered in the rest of this article. Since the system of first-order equations [(8) and (9)] is autonomous for constant $\dot{\epsilon}$, we can eliminate the variable t and obtain

$$\frac{d\sigma^*}{d\sigma_a} = \frac{DC\sigma^* (\sigma_a/\sigma^*)^k (\ln \sigma^*/\sigma_a)^{-l}}{\mathcal{M}[\dot{\epsilon} - D(\sigma^*/G)^m (\ln \sigma^*/\sigma_a)^{-l}]}. \tag{10}$$

Numerical difficulties in integrating eqn (10) or eqns (8) and (9) arise in the regime of low temperature or very high strain rates. The material parameters D, C, m, l and k for pure nickel at room temperature are $D = 6.4 \times 10^{-23}$, $C = 9.0 \times 10^{-9}$, $m = 5$, $l = 6.6$ and $k = 7$ (K. C. Wu, personal communication). The stiffness of eqn (10) or eqn (8) and (9) is clear if we assume the nonelastic strain rate to be 10^{-4}. The extremely small values of D and C imply that there is almost no plastic straining until σ_a is extremely close to σ^*. At this point, the numerical procedure breaks down as the logarithmic term in eqn (10) becomes extremely large. This numerical instability is avoided by various investigators[3, 4] using the viscoplastic limit proposed by Van Arsdale *et al.*[5]. The basic idea behind the viscoplastic limit is that σ_a is assumed to stay very close to σ^* if the plastic strain rate $\dot{\alpha}$ is slightly greater than zero. Thus, in the numerical procedure, σ_a is set to be equal to σ^* when σ_a reaches a certain critical fraction of σ^*, say $|\sigma_a - \sigma^*|/\sigma^* \approx 0.001$. In general, there is no systematic way of choosing the critical value of this fraction. Furthermore, the choice of $\sigma_a = \sigma^*$ clearly violates the system of differential equations [(8) and (9) or (10)]. Thus, the usual numerical procedure is not to integrate the full system of eqns (1), (3), (5) and (7), but instead to integrate only the system of equations defined by eqns (1), (3) and (7a), which do not contain the term $(\ln \sigma^*/\sigma_a)^{-l}$ explicitly. So far, no rigorous justifications have been given for the use of the viscoplastic limit (e.g. it is not clear which equations of the system should be retained and what the effect of the critical fraction is on the accuracy of the numerical result). Various investigators have attempted to devise new numerical schemes to integrate this system of equations without using the viscoplastic limit. These schemes are currently under development and very few results are yet available.

ANALYSIS OF GOVERNING EQUATIONS

In this section, we will solve eqn (10) in the case of low temperature (or very high strain rate) using the method of matched asymptotics. At high temperatures, the governing equations are not stiff enough to present any numerical problems[4]. This is because D is a very sensitive function of temperature [eqn (6)].

The first step is the nondimensionalization of eqn (10) or eqns (8) and (9). The following nondimensional variables are introduced:

$$y = \frac{\sigma^*}{\sigma_0^*}, \quad x = \frac{\sigma_a}{\sigma_0^*}, \quad \tau = \frac{\dot{\epsilon} t}{(\sigma_0^*/\mathcal{M})}, \tag{11}$$

where σ_0^* is defined by the initial condition $\sigma^*(t = 0) = \sigma_0^*$. With these new nondimensional variables, eqns (8) and (9) become

$$\frac{dy}{d\tau} = \epsilon_2 y \left(\frac{x}{y}\right)^k \left(\ln \frac{y}{x}\right)^{-l} \tag{12}$$

$$\frac{dx}{d\tau} = 1 - \epsilon_1 y^m \left(\ln \frac{y}{x}\right)^{-l} \tag{13}$$

with

$$\epsilon_1 = \frac{D}{\dot{\epsilon}} \left(\frac{\sigma_0^*}{G}\right)^m \tag{14}$$

and

$$\epsilon_2 = \frac{DC}{\dot{\epsilon}} \left(\frac{\sigma_0^*}{\mathcal{M}}\right). \tag{15}$$

Since eqns (12) and (13) are a system of autonomous differential equations, we may eliminate the variable τ and obtain the following:

$$\frac{dy}{dx} = \frac{\epsilon_2 y\,(x/y)^k\,(\ln y/x)^{-l}}{1 - \epsilon_1 y^m\,(\ln y/x)^{-l}} \tag{16}$$

with the initial condition

$$y(x = 0) = 1. \tag{17}$$

The assumption of low temperature or high strain rate implies that the conditions

$$\epsilon_2 \ll 1, \quad \epsilon_1 \ll 1 \tag{18}$$

are satisfied. For the case of nickel at room temperature, $\epsilon_1 = 2.05 \times 10^{-32}$ and $\epsilon_2 = 1.23 \times 10^{-29}$. The equivalence of high strain rate and low temperature is clearly seen from eqns (14) and (15). Notice that the term D is solely responsible for the temperature dependence of ϵ_1 and ϵ_2. As D can change by many orders of magnitude, dependent on the temperature change, it plays a much stronger role in determining the stiffness of these equations than the nonelastic strain rate. For example, for nickel at room temperature, changing $\dot{\epsilon}$ from 10^{-4} to 10^{-10} would not change the stiffness of eqn (16) appreciably. The advantage of this nondimensionalization is now clear; by a judicious choice of these parameters, the validity of the viscoplastic limit can be examined numerically. This approach will be pursued in the next section. We will now present the uniform asymptotic solution of eqn (16) for all practical ranges of $1 \leq y \leq 10^2$.

The form of eqn (16) suggests that for very small ϵ_1 and ϵ_2, y will be practically independent of x (i.e. $y \simeq 1$) until x approaches 1. As x approaches 1, we anticipate an abrupt change in the behavior of y (i.e. the material starts hardening and nonelastic flow occurs). We also expect that after this abrupt change, x remains close to y as y increases beyond 1. We therefore anticipate a mathematical internal boundary layer located approximately at $x = 1$. We will first analyze the behavior of y (i.e. one of the outer solutions) for $x < 1$. We begin by assuming a regular perturbation expansion of the form

$$y = 1 + \epsilon_2 y_2 + \epsilon_1 y_1 + \text{terms of order higher than } \epsilon_2 \text{ or } \epsilon_1, \tag{19}$$

where y_2 and y_1 are assumed to be of the order 1. Substituting this into eqn (16) and matching order in ϵ_1 and ϵ_2, it is easy to verify that

$$y_1 = 0 \tag{20}$$
$$y_2 = \int_0^x \beta^k(-\ln \beta)^{-l}\, d\beta.$$

Note that y_2 satisfies the condition $y_2(x = 0) = 0$. This has to be the case as $y(0) = 1$. Note also that the effect of ϵ_1 matters only if one seeks higher-order terms in the asymptotic series eqn (19). Thus, for $x < 1$,

$$y \sim 1 + \epsilon_2 \int_0^x \beta^k(-\ln \beta)^{-l}\, d\beta + \text{higher-order terms}. \tag{21}$$

As mentioned previously, eqn (21) cannot be uniformly valid for all values of x. Equation (21) implies that as $x \to 1$, the assumption that y_2 is of the order 1 will no longer be correct. The region about $x = 1$ in which the outer solution is not valid can be estimated by solving

$$\epsilon_2 y \left(\frac{x}{y}\right)^k \left(\ln \frac{y}{x}\right)^{-l} = 1, \tag{22}$$

assuming $y \sim 1$ and $x = 1 - \delta$. To the first order, δ is found to be $\epsilon_2^{1/l}$. We now define the stretching variables $\bar{x}$ and $\bar{y}$:

$$\bar{x} = \frac{x - 1}{\epsilon_2^{1/l}} \tag{23}$$

$$\bar{y} = \frac{y - 1}{\epsilon_2^{1/l}}. \tag{24}$$

Equation (24) implies that we are interested only in first-order solutions.

The estimation of the boundary layer [eqn (22)] has made use of the assumption that $\epsilon_1 < \epsilon_2$. A more rigorous justification is to use the concept of distinguished limit, in which $\epsilon_1 = \epsilon_2^p$ for some positive p. If $p > 1$, then $\epsilon_1 \ll \epsilon_2$. If $0 < p < 1$, then $\epsilon_1 \gg \epsilon_2$. Thus, for the case of $p > 1$, we anticipate the thickness of the boundary layer is of the order $\delta_2 \equiv \epsilon_2^{1/l}$ [using eqn (22)]. For the case of $0 < p < 1$, we expect the boundary layer thickness to be given by $\epsilon_2^{p/l}$ or by $\epsilon_1^{1/l} \equiv \delta_1$. This is because the boundary layer thickness is now estimated by solving

$$1 - \epsilon_2^p y^m \left(\ln \frac{y}{x}\right)^{-l} \sim 0 \tag{25}$$

to the first order.

A change of the dependent variable proves to be useful at this point, if we let

$$z = \frac{y}{x}. \tag{26}$$

Equation (16) becomes

$$z + x\frac{\mathrm{d}z}{\mathrm{d}x} = \frac{\epsilon_2 x z^{-k+1}(\ln z)^{-l}}{1 - \epsilon_1 x^m z^m (\ln z)^{-l}}. \tag{27}$$

The regular perturbation series [eqn (21)] for $x < 1$ becomes, to the first order,

$$z \sim \frac{1}{x} + \frac{\epsilon_2}{x}\int_0^x \beta^k(-\ln\beta)^{-l}\,\mathrm{d}\beta. \tag{28}$$

Consistent with eqns (23) and (24), we define the stretch variables $\tilde{x}$ and $\tilde{z}$ by

$$z = 1 + \epsilon_2^{1/l}\tilde{z} = 1 + \delta_2\tilde{z} \tag{29}$$

$$x = 1 + \epsilon_2^{1/l}\tilde{x} = 1 + \delta_2\tilde{x} \tag{30}$$

for the case of $p \geq 1$. Substituting eqns (29) and (30) into eqn (23), keeping terms no higher than first order and using the fact that $\ln(1 + \beta) = \beta$ for small positive β, we have

$$1 + \frac{\mathrm{d}\tilde{z}}{\mathrm{d}\tilde{x}} = \frac{\tilde{z}^{-l}}{1 - r\tilde{z}^{-l}} \tag{31}$$

with

$$r = \frac{\epsilon_1}{\epsilon_2} = \frac{(\sigma_0^*/G)^{m-1}(\mathcal{M}/G)}{C}.$$

The solution of eqn (31) is

$$\tilde{z} - \int_{\tilde{z}}^{\infty}\frac{\mathrm{d}\beta}{\beta^l - b_2} = -\tilde{x} + \text{constant}, \tag{32}$$

where b_2 is defined to be

$$b_2 = 1 + \frac{\epsilon_1}{\epsilon_2}. \tag{33}$$

Note that since l is always much greater than 1, there is no convergence problem with the integral

$$\int_{\tilde{z}}^{\infty}\frac{\mathrm{d}\beta}{\beta^l - b_2}.$$

The constant in eqn (32) has to be determined by matching with the outer solution [eqn (28)]. Assuming there is a matching region, we expand the outer solution [eqn (28)] in terms of $\tilde{z}$ and $\tilde{x}$, and the result is

$$\tilde{z} \sim -\tilde{x} - \frac{1}{l-1}(-\tilde{x})^{-l+1} + \text{higher-order terms}. \tag{34}$$

Matching requires that as $\tilde{x} \to -\infty$, eqns (34) and (32) agree to the same order. Thus, the constant term in eqn (32) must be set to zero. $\tilde{z}(\tilde{x})$ is now completely determined by

$$\tilde{z} - \int_{\tilde{z}}^{\infty} \frac{d\beta}{\beta^l - b_2} = -\tilde{x}. \tag{35}$$

It is easy to verify that in the other limit, where $\epsilon_1 = \epsilon_2^p$ and $0 < p < 1$, the internal boundary layer solution $\tilde{z}$ defined by

$$\begin{aligned} z &= 1 + \epsilon_1^{1/l}\,\tilde{z} \\ x &= 1 + \epsilon_1^{1/l}\,\tilde{x} \end{aligned} \tag{36}$$

is given implicitly by

$$\tilde{z} - \frac{\epsilon_2}{\epsilon_1} \int_{\tilde{z}}^{\infty} \frac{d\beta}{\beta^l - b_1} = -\tilde{x}, \tag{37}$$

where

$$b_1 = 1 + \frac{\epsilon_2}{\epsilon_1}. \tag{38}$$

We next consider the outer solution in the region $x > 1$. Here, we anticipate that

$$z = 1 + \rho(x, \epsilon_1, \epsilon_2) \tag{39}$$

and that $d\rho/dx \to 0$ as $\epsilon_1, \epsilon_2 \to 0$. Using these assumptions, we substitute eqn (39) into (27) and find to the first order

$$z \sim 1 + (\epsilon_2 x + \epsilon_1 x^m)^{1/l}. \tag{40}$$

Note that dz/dx indeed goes to 0 as $\epsilon_1, \epsilon_2 \to 0$ for any finite x. The uniformity of the asymptotic result is ensured by

$$\begin{aligned} z(x \to 1) &= \lim_{x \to 1} [1 + (\epsilon_2 x + \epsilon_1 x^m)^{1/l}] \\ &= 1 + (\epsilon_2 + \epsilon_1)^{1/l} \\ &= 1 + \epsilon_i^{1/l} (\lim_{\tilde{x} \to \infty} \tilde{z}), \qquad i = 1, 2. \end{aligned}$$

The above equation implies that the internal boundary layer solution matches with the outer solution for $x > 1$. To summarize, the uniform approximate solution for eqn (17) is

$$y \sim 1 + \epsilon_2 \int_0^x \beta^k (-\ln \beta)^{-l}\, d\beta, \qquad x < 1 \tag{41}$$

$$y \sim x + x(\epsilon_1 x^m + \epsilon_2 x)^{1/l}, \qquad x > 1. \tag{42}$$

In the boundary layer, the solution $\tilde{y}$ is given implicitly by

$$\tilde{z} - \lambda_i \int_{\tilde{z}}^{\infty} \frac{d\beta}{\beta^l - b_i} = -\tilde{x}, \tag{43}$$

where $i = 1$ if $0 < p < 1$, $i = 2$ if $p > 1$, $\lambda_1 = 1$ and $\lambda_2 = \epsilon_2/\epsilon_1$. $\bar{y}$ is related to $\bar{z}$ and $\bar{x}$ by eqns (23), (24) and (26). To the first order, this is

$$\bar{y} = \bar{z} + \bar{x}. \tag{44}$$

The asymptotic results above imply that the internal boundary layer has a thickness of the order of $\delta = \max(\epsilon_2^{1/l}, \epsilon_1^{1/l})$. For pure nickel at room temperature with an applied nonelastic strain rate $\dot{\epsilon} = 10^{-4}$, $\delta \approx 0(10^{-5})$. For our assumed values of ϵ_1 and ϵ_2, eqn (42) implies $y \simeq x + 0(10^{-4})$ or $\sigma^* = \sigma_a + 0(10^{-4})\,\sigma_a$ once yielding occurs. Thus, the error made by the viscoplastic limit approximation (i.e. by setting $\sigma_a = \sigma^*$) is less than several psi, assuming that $\sigma_0^* = 2.2 \times 10^4$ psi at 25°C. For the case of nickel at room temperature, it is clear that we could treat σ^* as independent of σ_a until σ_a reached $\sigma_0^* - 0(\delta)$. For $\sigma_a \geq \sigma_0^* - 0(\delta)$, we can use the approximation $\sigma^* = \sigma_a$. Numerically, the plastic strain rate $\dot{\alpha}$ can be integrated by using eqn (7a), i.e.

$$\dot{\sigma}^* = C\sigma^* \left(\frac{\sigma_a}{\sigma^*}\right)^k \left(\frac{G}{\sigma^*}\right)^m \dot{\alpha}. \tag{45}$$

Using $(\sigma_a/\sigma^*) = 1 + 0(\delta) \approx 1$, we have

$$\alpha = \frac{1}{mC}\left(\frac{\sigma^*}{G}\right)^m + \text{constant}. \tag{46}$$

If $\delta \ll 1$, as in nickel, the constant in eqn (46) can be determined approximately by the condition $\alpha = 0$ when $\sigma_a = \sigma_0^*$. Using $\sigma_a \sim \sigma^*$ for $\sigma_a > \sigma_0^*$, eqn (46) becomes

$$\alpha = \frac{1}{mC}\left[\left(\frac{\sigma_a}{G}\right)^m - \left(\frac{\sigma_0^*}{G}\right)^m\right], \qquad \sigma_a > \sigma_0^*. \tag{47}$$

For $\sigma_a \leq \sigma_0^*$, we have, to the order of δ,

$$\alpha = 0, \qquad \sigma_a \leq \sigma_0^*. \tag{48}$$

The above analysis suggests that eqns (47) and (48) remain a good approximation for more general loading histories as long as there is no unloading. They imply that Hart's equation for the plastic strain α is equivalent to that of a classical deformation material (where stress is a point function of strain), as long as one is willing to accept errors of the order of δ. From the above discussion, it is clear that the maximum relative error made using eqns (47) and (48) is in the transition region. The relative error reduces appreciably as the material strain hardens, i.e. for $\sigma^* \gg \sigma_0^*$.

We also note that the outer solution [eqn (42)] for $x > 1$ is accurate as long as $(\epsilon_1 x^m + \epsilon_2 x)^{1/l} \ll 1$. It is expected that this condition is generally satisfied for all realistic values of x at low temperatures. For example, for nickel at 25°C and for all $x < 10^2$ (or $\sigma_a < 10^2\sigma_0^* \sim G/10$), $(\epsilon_1 x^m + \epsilon_2 x)^{1/l} \leq 10^{-4}$.

NUMERICAL RESULTS

Numerical results will be presented in this section to support the analytical results derived in the preceding section. The system of eqns (1), (3), (5) and (7) is integrated numerically without using the viscoplastic limit approximation. The choice of ϵ_1, ϵ_2 is such that their values are sufficiently small to reflect the features of the viscoplastic limit approximation but still large enough so that the equations can be integrated without a substantial amount of numerical difficulty.

The results of the previous section indicate that for sufficiently small values of ϵ_1 and ϵ_2, the thickness of the transition region is of the order of $\delta = \max(\epsilon_1^{1/l}, \epsilon_2^{1/l})$. Thus, suitable choices for our numerical experiment are

$$10^{-2} \leq \epsilon_2^{1/l} \ll 1, \qquad 10^{-2} \leq \epsilon_2^{1/l} \ll 1.$$

A program developed by Mukherjee[4] is used to integrate these equations without employing the viscoplastic limit approximation for the case of $\epsilon_1 = 10^{-10}$ and $\epsilon_2 = 10^{-9}$. For convenience, instead of $l = 6.6$, $l = 6$ is used in the numerical computation. Nondimensional plots of σ^* vs. σ_a are shown in Figs. 1 and 2 for these parameters. The viscoplastic limit approximation is also included in Fig. 1 as a comparison. In Fig. 2, the scale is chosen so that the transition region can be examined in detail. The outer solutions for $x < 1$ and $x > 1$ [i.e. eqns (41) and (42)] are also included in this figure. Figure 2 clearly indicates that for $x < 0.965$, the outer solution [eqn (41)] is practically identical to the numerical results. For $x > 0.975$, the outer solution [eqn (42)] is indistinguishable from the numerical solution. Note that if we extrapolate the numerical solution for $x > 0.98$ using a straight line, the value of the x intercept of this line is ~0.97. Thus, the distance of this intercept from the point $\sigma_a = \sigma_0^*$ or $x = 1$ is ~0.03 and is approximately equal to $\delta = 0.0316$. Figures 1 and 2 also show that although the values of ϵ_1 and ϵ_2 in this example are much larger than that of nickel at room temperature, the viscoplastic limit approximation gives a very reasonable estimate of the numerical result within error of the order of δ. The results presented in Fig. 2 show that the boundary layer solution [eqn (43)] matches extremely well with the outer solutions [eqns (41) and (42)]. The agreement of the boundary layer solution with the numerical result is also excellent. A composite solution is not possible for the entire range of values of x, as one of the outer solutions [eqn (41)] diverges as $x \to 1$. In the numerical integration, the maximum step size is determined by the condition that decreasing the step size does not change the values of all the dependent variables (including the rate quantities). For this set of parameter values, maximum step size of 10^{-4} satisfies this criterion. We notice, however, that increasing the maximum step size from 10^{-3} to 10^{-2} changes the values of the rate quantities substantially (e.g. $\dot{\alpha}$), although no practical changes are observed for the variables α, σ^* and σ_a.

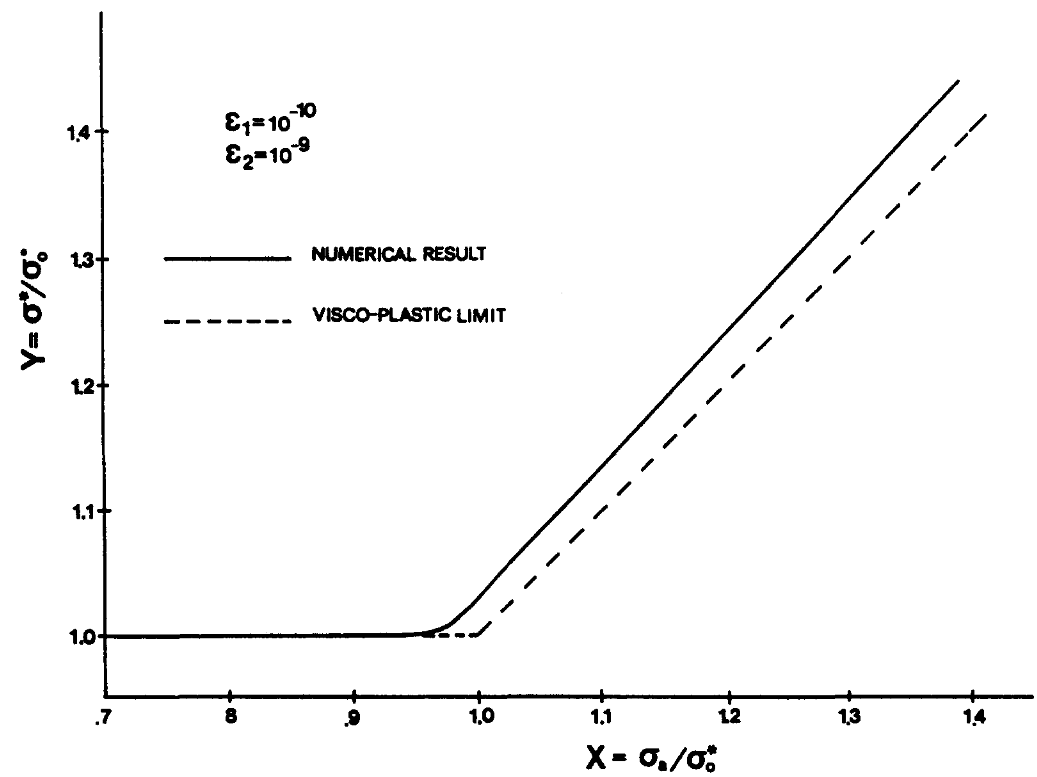

Fig. 1. Nondimensional plots of σ^* vs. σ_a.

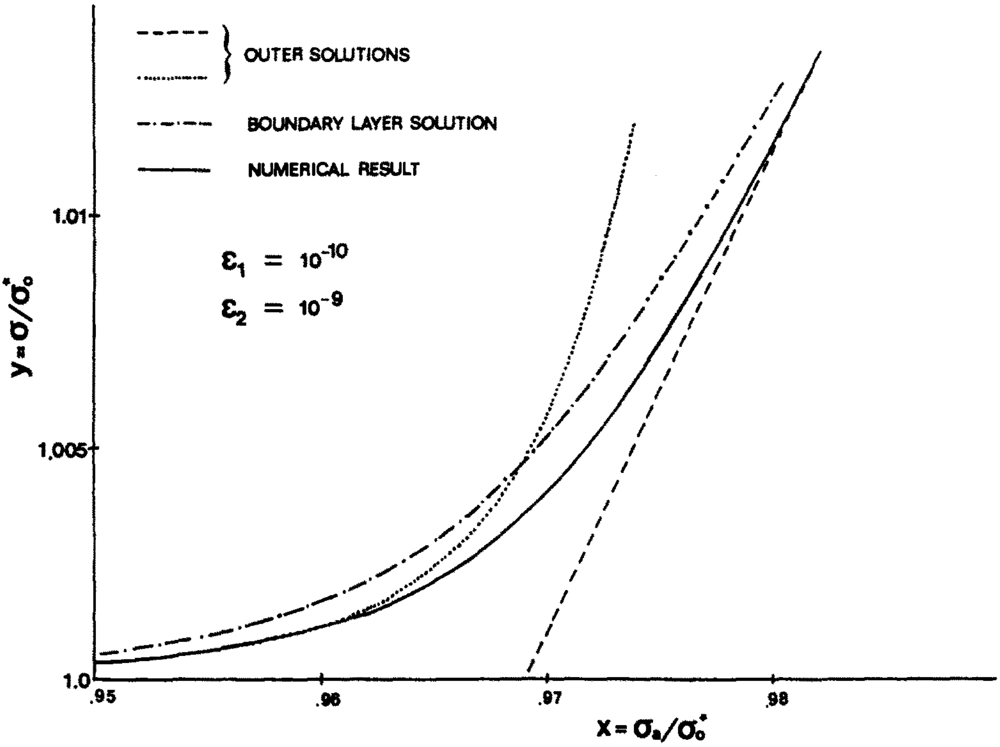

Fig. 2. Comparison of numerical and asymptotic results.

CONCLUSION

The analytical and numerical results presented in the previous sections clearly justify the viscoplastic limit approximation for the case of the tensile test for constant, nonelastic strain rate and low temperatures. The governing equations in this case depend on two nondimensional parameters, ϵ_1 and ϵ_2. The approximate behavior of the solution for $\sigma_a > \sigma_0^*$ (or $x > 1$) is

$$\sigma^* \approx \sigma_a + \sigma_a \left[\epsilon_1 \left(\frac{\sigma_a}{\sigma_0^*}\right)^m + \epsilon_2 \left(\frac{\sigma_a}{\sigma_0^*}\right) \right]^{1/l} .$$

At low temperature, ϵ_1 and ϵ_2 are generally small so that the second term is almost insignificant compared with the first for all practical values of σ_a, i.e. $\sigma_a/\sigma_0^* < 10^2$. The validity of the viscoplastic limit approximation can also be shown by examining the qualitative behavior of Hart's equations. This will be presented in future work. The uniformly valid asymptotic solution predicts very rapid changes of plastic strain rate in a small region of the order of $\delta = \max(\epsilon_1^{1/l}, \epsilon_2^{1/l})$ about $\sigma_a \approx \sigma_0^*$. The amount of accumulated plastic strain at this level of stress is only of the order of δ. Since, at low temperatures, the values of ϵ_1 and ϵ_2 are extremely small, a possible numerical procedure is to use the viscoplastic limit approximation when $\sigma_a = \sigma_0^* - 0(\delta)\,\sigma_0^*$. Our solution suggests that for low temperature applications, the plastic element in Hart's model can be treated approximately as a deformation element. The error made in using this approximation is of the order of δ.

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